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ON THE ZEROS OF POWER SERIES WITH HADAMARD GAPS—DISTRIBUTION IN SECTORS[1]

BY

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ABSTRACT. We give a sufficient condition for a power series with Hadamard gaps to assume every complex value infinitely often in every sector of the unit disk.

I. Introduction. Let

$$f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k} \tag{1}$$

be a power series convergent in $|z| < 1$, with Hadamard gaps, $n_{k+1}/n_k > q > 1$, $k \geq 1$. Given a complex number c, we are interested in the distribution of the zeros of $f(z) - c$. We shall discuss the problem in term of the zeros of f, replacing the constant term c_0 of (1) by $c_0 - c$ if necessary.

It has been shown that

(i) f has infinitely many zeros in the unit disk if $\Sigma_{k=0}^{k=\infty} |c_k| = \infty$ and $q \geq q_0$, where q_0 is about 100 [5].

(ii) f has infinitely many zeros in any sector $\theta_2 < \arg z < \theta_1$, $|z| < 1$, if $\limsup_{k\to\infty} |c_k| > 0$ [2].

It remains undetermined whether f has zeros in the unit disk, or perhaps in any sector, if $\Sigma_{k=0}^{k=\infty} |c_k| = \infty$, $\lim_{k\to\infty} c_k = 0$, and $1 < q < q_0$. We prove

Theorem 1. *Let $f(z) = c_0 + \Sigma_{k=1}^{\infty} c_k z^{n_k}$ be a power series convergent in $|z| < 1$, with*

(i) $n_{k+1}/n_k > q > 1$ $(k \geq 1)$,

(ii) $\lim_{k\to\infty} c_k = 0$,

(iii) $\Sigma_{k=0}^{\infty} |c_k|^{2+\epsilon} = \infty$ *for some positive* ϵ.

Then f has infinitely many zeros in any sector $\theta_2 < \arg z < \theta_1$, $|z| < 1$.

II. A formula. Basic to the proof of Theorem 1 is a formula for functions

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meromorphic in sectors. The basic idea of this formula goes back to V. P. Petrenko [3]. The following lemma can be found in [2].

Lemma 1. *Suppose $f(z)$ is meromorphic in the sector $|\arg z| \le \pi/\nu$ $(\nu > 1)$, $|z| \le R$. Let $z = t$ $(0 < t < R)$ be a regular point of f on the real axis, where $f(t) \ne 0$. For $z \ne t$, R^2/t, define*

$$a(z) = a(R, z, t) = \log |(R^2 - tz)/R(z - t)|$$

and

$$A(R, z, t) = a(z) - a(-|z|).$$

If we write

$$I_1(R, t, \nu) = \int_0^R \left\{ \int_{-\pi/\nu}^{\pi/\nu} \log |f(re^{i\theta})|\, d\theta \right\} \xi_1(R, r, t, \nu)\, dr,$$

$$I_2(R, t, \nu) = \int_{-\pi/\nu}^{\pi/\nu} \log |f(Re^{i\theta})| \xi_2(R, \theta, t, \nu)\, d\theta,$$

where

$$\xi_1(R, r, t, \nu) = \frac{\nu^2}{2\pi} \frac{r^{\nu-1}t^\nu(R^{2\nu} - t^{2\nu})(R^{2\nu} - r^{2\nu})}{(r^\nu + t^\nu)^2(R^{2\nu} + r^\nu t^\nu)^2},$$

$$\xi_2(R, \theta, t, \nu) = \frac{\nu}{\pi} \frac{R^\nu t^\nu (R^\nu - t^\nu)(1 + \cos \nu\theta)}{(R^\nu + t^\nu)(R^{2\nu} + t^{2\nu} - 2R^\nu t^\nu \cos \nu\theta)}$$

then

$$\log |f(t)| = I_1(R, t, \nu) + I_2(R, t, \nu) + \sum_{b_j} A(R^\nu, t^\nu, b_j^\nu) - \sum_{a_i} A(R^\nu, t^\nu, a_i^\nu) \tag{2}$$

where the summation is taken over the zeros $\{a_i\}$ and the poles $\{b_i\}$ of f which lie in the interior of the sector.

Without loss of generality, we may assume that $f(0) = 1$ (consider $f(z)/c_p z^p$ if necessary). Suppose now that f has no zero in some sector, which we may assume to be the sector $|\arg z| \le \pi/\nu_0$, $|z| < 1$, where $\nu_0 > 1$. We shall show that this leads to the conclusion

$$\limsup_{R \to 1} [I_1(R, 2\nu_0) + I_2(R, 2\nu_0)] = \infty \tag{3}$$

whereas (2) now reduces to the contradictory result

$$I_1(R, 2\nu_0) + I_2(R, 2\nu_0) = \log |f(t)|.$$

In the next section, we derive estimates which will be used to establish (3) in §IV.

III. Lower bounds for $|f(z)|$. Transform the domain of f to the right half-plane with the change of variable $z = e^{-w}$, and write (1) as

$$F(w) = f(e^{-w}) = c_0 + \sum_{k=1}^{\infty} c_k e^{-n_k w} \tag{4}$$

Lemma 2. *There exist a subsequence $\{c_{k(i)}\}$ of the coefficients $\{c_k\}$ of* (4) *and positive constants $U_0(q)$, $u_0(q)$, $p_0(q)$ such that the derivatives of F satisfy*

$$F^{(p)}(w) = (-n_{k(i)})^p c_{k(i)} e^{-n_{k(i)} w} + R_i(w),$$

$$|R_i(w)| \leq \tfrac{1}{2} |c_{k(i)}| n_{k(i)}^p e^{-n_{k(i)} \operatorname{Re}(w)}$$

whenever $p \geq p_0(q)$, and Re (w) *is in the range*

$$u_0(q)/n_{k(i)} < \operatorname{Re}(w) < U_0(q)/n_{k(i)}.$$

Proof. Consider the sequence $\{d_k\}$, where

$$d_0 = \max\{|c_0|, |c_1|, |c_2|, \cdots\},$$

$$d_k = \max\{\tfrac{1}{2} d_{k-1}, |c_k|, |c_{k+1}|, \cdots\} \qquad (k \geq 1),$$

one verifies readily that

(a) $d_k > 0$ for all k,

(b) $1/2 \leq d_{k+1}/d_k \leq 1$, and also

(c) $d_k \geq |c_k|$, with equality occuring infinitely often.

If in (c), equality occured finitely often, then $d_{k+1} = \tfrac{1}{2} d_k$ for $k \geq k_0$. In this case

$$\sum_{k=k_0}^{\infty} |c_k| \leq \sum_{k=k_0}^{\infty} d_k = d_{k_0} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j < \infty,$$

contradicting the assumptions that $\sum_{k=0}^{k=\infty} |c_k|^{2+\epsilon} = \infty$ and $\lim_{k\to\infty} c_k = 0$.

Let $\{c_{k(i)}\}$ be the subsequence of $\{c_k\}$ satisfying $d_{k(i)} = |c_{k(i)}|$, $i = 1, 2, \cdots$. Differentiating $F(w)\, p$ times, (4) becomes

$$F^{(p)}(w) = \sum_{k=1}^{\infty} \delta_k a_k(w)$$

where $\delta_k = (-1)^p c_k/d_k$, and $a_k(w) = (n_k)^p d_k \exp(-n_k w)$. We can find, for each $k(i)$, and for sufficiently large p, a set of w such that

$$|a_{k+1}(w)|/|a_k(w)| > 5 \qquad \text{for } k < k(i), \tag{5}$$

$$|a_{k+1}(w)|/|a_k(w)| < 1/5 \qquad \text{for } k \geq k(i). \tag{6}$$

For, (5) holds if

$$\mathrm{Re}(n_{k+1}w) < (p \log t_k - \log 10)/(1 - 1/t_k)$$

where $t_k = n_{k+1}/n_k > q$. For sufficiently large p,

$$f(t) = (p \log t - \log 10)/(1 - 1/t)$$

is a positive increasing function of t in $t \geq q$. Therefore, (5) holds if $p \geq p_0$, and if

$$\mathrm{Re}(w) < (1/n_{k(i)})(p \log q - \log 10)/(1 - 1/q).$$

Similarly (6) holds, if

$$\mathrm{Re}(n_k w) > (p \log t_k + \log 5)/(t_k - 1).$$

The right-hand side of this inequality is bounded above by $(p \log q + \log 5)/(q - 1)$, so that (6) holds if

$$\mathrm{Re}(w) > (1/n_{k(i)})(p \log q + \log 5)/(q - 1).$$

We note that if $u = (p \log q + \log 5)/(q - 1)$, and $U = (p \log q - \log 10)/(1 - 1/q)$, then for large p, $U/u = q(1 + O(1/p)) > c > 1$. Thus (5) and (6) hold simultaneously, if $p > p_1$, and $\mathrm{Re}(w)$ satisfies

$$u/n_{k(i)} < \mathrm{Re}(w) < U/n_{k(i)}. \tag{7}$$

If w is in the range of (7), we have then

$$F^{(p)}(w) = \delta_{k(i)} a_{k(i)}(w) + \sum_{k \neq k(i)} \delta_k a_k(w)$$

$$= \delta_{k(i)} a_{k(i)}(w) + R_i(w)$$

where $|\delta_{k(i)}| = 1$, and

$$|R_i(w)| \leq \sum_{1 \leq k < k(i)} |a_{k(i)}(w)|(5)^{k-k(i)} + \sum_{k > k(i)} |a_{k(i)}(w)|(5)^{k(i)-k}$$

$$\leq 2|a_{k(i)}(w)| \sum_{j=1}^{\infty} (5)^{-j} = \frac{|a_{k(i)}(w)|}{2}.$$

Lemma 3. *Let $F(w)$ be holomorphic in $|w - w_0| \leq R$. If for some p, $|F^{(p)}(w)| \geq m > 0$ and $\sup_{|w-w_0| \leq R} |F^{(p)}(w)| = M$, then the image of $|w - w_0| < R$ under F covers the disk*

$$\{z\colon |z - F(w_0)| < K_p R^p m^{p+1} M^{-p}\}$$

where K_p is a positive constant depending on p only [1].

We infer from Lemma 2 and Lemma 3 the following

Lemma 4. *If the function f of Theorem 1 has no zero in the sector $|\arg z| \leq \pi/\nu_0$, $|z| < 1$, then there exist positive constants U_1, u_1, and L, depending on q only, such that $|f(z)| > L\,|c_{k(i)}|$ in*

$$S_i\colon \exp(-U_1/n_{k(i)}) < |z| < \exp(-u_1/n_{k(i)}),$$

$$|\arg z| < \pi/\nu \qquad (\nu = 2\nu_0).$$

Here $\{k(i)\}$ is the sequence defined by $|c_{k(i)}| = d_{k(i)}$.

We next estimate the size of the set of points where

$$|f(z)|\left\{\frac{1}{2}\left(|c_0|^2 + \sum_{k=1}^{\infty} |c_k|^2|z|^{2n_k}\right)\right\}^{-1/2}$$

is bounded away from zero. The following result is due to R. Salem and A. Zygmund. The basic idea of the proof can be found in [4]. Define

$$A(r) = \left\{\frac{1}{2}\left(|c_0|^2 + \sum_{k=1}^{\infty} |c_k|^2 r^{2n_k}\right)\right\}^{1/2}.$$

Lemma 5. *If f satisfies the conditions of Theorem 1, then, in any measurable subset $E \subset [0, 2\pi]$, the linear measure*

$$m\{\theta \in E \mid |f(re^{i\theta})|A(r)^{-1} \leq y\}$$

tends to $(m(E)/2\pi)\int_0^{2\pi}\int_0^y re^{-r^2/2}\,dr = m(E)(1 - e^{-y^2/2})$ as $r \to 1$.

Lemma 6. *For any measurable subset $E \subset [0, 2\pi]$, and any positive $\delta < 1$, there is r_0 such that whenever $r \geq r_0$,*

(8) $$m\{\theta \in E \mid |f(re^{i\theta})|A^{-1}(r) > \delta\} \geq m(E)(1 - \delta).$$

Proof. By Lemma 5, for $r < 1$,

$$m\{\theta \in E \mid |f(re^{i\theta})|A^{-1}(r) \leq y\} = m(E) - \{\theta \in E \mid |f(re^{i\theta})|A^{-1}(r) > y\}$$

$$\to m(E)e^{-\frac{1}{2}y^2} \qquad (r \to 1).$$

Set $y = \delta$. Since $\exp(-\delta^2/2) > 1 - \delta^2/2 > 1 - \delta$, (8) is proved.

IV. Lower bounds for $I_1(R) + I_2(R)$. In the following derivations, we shall use letters $K_1, K_2, K_3, \cdots$ for positive constants which depend on f, t and ν, but not on R.

With the notations of Lemma 1,

$$I_2(R, t, \nu) \geq \int_{-\pi/2\nu}^{\pi/2\nu} \log^+|f(Re^{i\theta})|\xi_2 d\theta - \int_{-\pi/\nu}^{\pi/\nu} \log^+|1/f(Re^{i\theta})|\xi_2\, d\theta.$$

In the first integral of the right-hand side $\xi_2 \geq K_1$ for all R sufficiently close to 1. Choose δ in the interval $0 < \delta < 1/2$. By Lemma 6, if $R \in S_i$ $(i \geq i_0)$, $\log^+|f| \geq \log A(R) + \log \delta$ in a subset of measure $> \pi/2\nu$ of $(-\pi/2\nu, \pi/2\nu)$.

In the second integral $0 \leq \xi_2 \leq K_2$. By Lemma 6, $\log^+|1/f(Re^{i\theta})| = 0$ outside a set of θ of measure $< K_3\delta$. In this set, by Lemma 4, $\log^+|1/f(Re^{i\theta})| < -\log(L|c_{k(i)}|)$.

Therefore, for all large i and $R \in S_i$,

$$
\begin{aligned}
I_2 &\geq K_4 \log A(R) + K_4 \log \delta + K_5\delta \log|c_{k(i)}| - K_6 \\
&\geq K_7 \log A(R) + K_4 \log \delta + K_5\delta \log|c_{k(i)}|.
\end{aligned} \tag{9}
$$

Next we find a lower bound for $I_1(R, t, \nu)$. From Lemma 1,

$$I_1 = 2\pi \int_0^R \xi_1(R, r, t, \nu)\left\{\frac{1}{2\pi}\int_{-\pi/\nu}^{\pi/\nu} \log|f(re^{i\theta})|\, d\theta\right\} dr$$

and we see that ξ_1 satisfies $0 \leq \xi_1 \leq K_8(R - r)$. By the first fundamental theorem of Nevanlinna,

$$\frac{1}{2\pi}\int_{-\pi/\nu}^{\pi/\nu} \log^+|1/f(re^{i\theta})|\, d\theta \leq T(r, f) = m(r, f).$$

By the inequality of the arithmetic and geometric mean

$$m(r, f) \leq K_9 \log A(r) \leq K_9 \log A(R) \qquad (r \leq R).$$

Therefore, if $0 < s < R$,

$$
\begin{aligned}
I_1 &\geq 2\pi \int_0^s \xi_1\left\{\frac{1}{2\pi}\int_{-\pi/\nu}^{\pi/\nu} \log|f(re^{i\theta})|\, d\theta\right\} dr \\
&\quad - 2\pi \int_s^R \xi_1\left\{\frac{1}{2\pi}\int_{-\pi/\nu}^{\pi/\nu} \log^+|1/f(re^{i\theta})|\, d\theta\right\} dr \\
&\geq D(s) - K_{10}\int_s^R (R - r)A(R)\,dr \geq D(s) - K_{11}(R - s)^2 A(R).
\end{aligned} \tag{10}
$$

By choosing s sufficiently close to 1, we can make

$$K_7 - K_{11}(R - s)^2 > K_7 - K_{11}(1 - s)^2 > \tfrac{1}{2}K_7.$$

Combining (9) and (10)

$$I_1 + I_2 \geq D(s_0) + \tfrac{1}{2}K_7 \log A(R) + K_5\delta \log|c_{k(i)}| + K_4 \log \delta \qquad (s_0 < R).$$

Since $R \in S_i$, $A(R) \geq K_{12} \sum_{k=0}^{k(i)} |c_k|^2$, and thus

$$I_1(R) + I_2(R) \geq K_{13}\left\{\log\left(|c_{k(i)}|^{\delta} \sum_{k=0}^{k(i)} |c_k|^2\right) + \log \delta\right\}.$$

To show that

(11) $$\limsup_{R\to 1}\,[I_1(R)+I_2(R)]=\infty$$

it is therefore enough to show that for some δ,

(12) $$\limsup_{i\to\infty}|c_{k(i)}|^{\delta}\left(\sum_{k=0}^{k(i)}|c_k|^2\right)=\infty.$$

We prove first that if $0<\delta<\epsilon/2$ where ϵ is the exponent of condition (iii) of Theorem 1, then

$$W(\delta)=\limsup_{p\to\infty}|c_p|^{\delta}\left(\sum_{k=0}^{p}|c_k|^2\right)$$

is infinite.

Suppose $W(\delta)<\infty$, then for some $K>0$, and all c_p with $|c_p|<1$,

(13) $$|c_p|^{2+\epsilon}\le|c_p|^{2+2\delta}\le K|c_p|^2\Big/\left(\sum_{k=0}^{p}|c_k|^2\right)^2.$$

Summing (13) over p,

$$\sum_{p=p_0}^{\infty}|c_p|^{2+\epsilon}\le K\sum_{p=0}^{\infty}\left\{|c_p|^2\Big/\left(\sum_{k=0}^{p}|c_k|^2\right)^2\right\}.$$

The left-hand side of the inequality is infinite by assumption. The right-hand side is finite by a well-known theorem on divergent series, stating that if $a_n\ge 0$, and $\sum_{n=0}^{n=\infty}a_n=\infty$, then for any positive ρ,

$$\sum_{p=0}^{\infty}\left\{a_p\Big/\left(\sum_{n=0}^{p}a_n\right)^{1+\rho}\right\}<\infty.$$

$W(\delta)$ must therefore be infinite.

Let $S=|c_p|^{\delta}(\sum_{k=0}^{k=p}|c_k|^2)$. We now prove (12) by showing that for at least one of the members of the sequence $\{k(i)\}$ which are closest to p,

$$|c_{k(i)}|^{\delta}\left(\sum_{k=0}^{k(i)}|c_k|^2\right)>\frac{2S}{3}.$$

The case $p\in\{k(i)\}$ is trivial. Suppose that $K<p$, and $K'>p$ are the two member of $\{k(i)\}$ which are closest to p. If, for some k in $K<k\le p$, $d_k=|c_l|$ $(l>k)$, then $l\in\{k(i)\}$, and by the definition of K and K', we must have $l=K'$ and $|c_p|<|c_{K'}|$, so that

$$|c_{K'}|^\delta \left(\sum_{k=0}^{K'} |c_k|^2 \right) > |c_p|^\delta \left(\sum_{k=0}^{p} |c_k|^2 \right) = S.$$

The only other possibility is that $d_k = \frac{1}{2} d_{k-1}$ $(K < k \leq p)$ and so $|c_k| \leq 2^{-k+K} d_K = 2^{-k+K} |c_K|$,

$$\sum_{k=0}^{p} |c_k|^2 \leq \sum_{k=0}^{K} |c_k|^2 + |c_K|^2 (\tfrac{1}{4} + (\tfrac{1}{4})^2 + \cdots),$$

$$\sum_{k=0}^{K} |c_k|^2 \geq \sum_{k=0}^{p} |c_k|^2 \left(1 - \frac{1}{3} \frac{|c_K|^2}{\sum_{k=0}^{p} |c_k|^2} \right) \geq \frac{2}{3} \sum_{k=0}^{p} |c_k|^2$$

if p is so large that $|c_K| < 1$, $\sum_{k=0}^{k=p} |c_k|^2 > 1$. We have now

$$|c_K|^\delta \left(\sum_{k=0}^{K} |c_k|^2 \right) \geq \frac{2}{3} |c_p|^\delta \left(\sum_{p=0}^{p} |c_k|^2 \right) = \frac{2S}{3}.$$

This proves (11) and completes the proof of Theorem 1.

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